On harmonic and asymptotically harmonic homogeneous spaces

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Abstract

We classify noncompact homogeneous spaces which are Einstein and *asymptotically harmonic*. This completes the classification of Riemannian *harmonic spaces* in the homogeneous case: Any simply connected homogeneous harmonic space is flat, or rank-one symmetric, or a nonsymmetric Damek-Ricci space. Independently, Y. Nikolayevsky has obtained the latter classification under the additional assumption of nonpositive sectional curvatures [Ni2].

1 Introduction

A complete Riemannian manifold (M, g) is called a *harmonic space* if about any point the geodesic spheres of sufficiently small radii are of constant mean curvature. If M is noncompact and harmonic, then it is also *asymptotically harmonic*, that is, M has no conjugate points and the mean curvature of its horospheres is constant (see Def. 2.1).

Clearly, any two-point homogeneous space is harmonic. In 1944, A. Lichnerowicz asked the central question in this field: Is a harmonic space M necessarily locally two-point homogeneous, that is, rank-one locally symmetric or flat?

Positive answers were given in 1944 by A. Lichnerowicz for $\dim M = 4$ [Li], in 1990 by Z. Szabó for compact M with finite fundamental group [Sz], in 1995 by G. Besson, G. Courtois, S. Gallot for compact M of negative sectional curvatures [BCG] (as a corollary of their proof of Gromov's minimal entropy conjecture, involving also results from [BFL] and [FL]), in 2002 by A. Ranjan

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and H. Shah for noncompact M with minimal horospheres [RS] (involving an observation of [Ni1]), and in 2002 by Y. Nikolayevsky [Ni1] for $\dim M = 5$.

However, in 1992, E. Damek and F. Ricci exhibited a class of noncompact harmonic spaces with sectional curvatures $K \leq 0$ which are *homogeneous* but nonsymmetric [DR].

In this class, only the rank-one symmetric spaces have strictly negative curvature [La], [Do2]. Any Damek-Ricci space admits a simply transitive, solvable Lie group of isometries S whose commutator subgroup $[S, S]$ is 2-step nilpotent and of codimension 1 in S. Conversely, any *harmonic* homogeneous space modeled on such a Lie group S is a Damek-Ricci space (see [Dr2], [BPR]).

It is therefore natural to consider the following problems:

(a) Are simply connected harmonic spaces necessarily homogeneous?

(b) Classify all homogeneous harmonic spaces.

Question (a) appears to be widely open.

Recently, (b) was solved by Y. Nikolayevsky [Ni2] under the additional assumption of nonpositive sectional curvatures.

In this paper, we give an independent solution of (b) which does not require any a priori curvature assumption. In fact, we prove the following more general statement:

Theorem 1.1. *Let M be a noncompact, simply connected homogeneous space. Then the following are equivalent:*

(i) *M is* asymptotically harmonic *and Einstein.*

(ii) *M is flat, or rank-one symmetric of noncompact type, or a nonsymmetric Damek-Ricci space.*

We also obtain some algebraic restrictions without the assumption that M is Einstein, i. e. of constant Ricci curvatures, see section 2.1.

Recall however, that every harmonic space is an Einstein manifold. Hence, Theorem 1.1 completes the classification of simply connected harmonic spaces in the homogeneous case:

Corollary 1.2. *Let M be a simply connected,* homogeneous harmonic space *of Ricci curvature c. Then, up to scaling of the metric, M is isometric to*

(a) S^n, $\mathbb{C}P^k$, $\mathbb{H}P^l$, *or* $\mathbb{O}P^2$, *if* $c > 0$.

(b) $\mathbb{R}^n$, *if* $c = 0$.

(c) $\mathbb{R}H^n$, $\mathbb{C}H^k$, $\mathbb{H}H^l$, $\mathbb{O}H^2$, *or a nonsymmetric Damek-Ricci space, if* $c < 0$.

In fact, if $c > 0$ holds, then it follows from the Bonnet-Myers Theorem and Z. Szabó's Theorem [Sz] that M is rank-one symmetric of compact type. The case of $c = 0$ is settled, since homogeneous Ricci flat spaces are necessarily flat [AlK]. Any homogeneous Einstein space with $c < 0$ is noncompact by a classical Bochner argument. If the space is harmonic in addition, then it has no conjugate points, is asymptotically harmonic (and Einstein) and Theorem 1.1 applies directly.

Recalling from Corollary 1 of [Heb3] (which is essentially based on [AlC]) that nonsymmetric Damek-Ricci spaces do not admit any quotients of finite volume, we obtain

Corollary 1.3. *Let M be a locally homogeneous, harmonic space of finite volume. Then, M is rank-one locally symmetric or flat.*

Note that Riemannian products of nonflat harmonic spaces are not harmonic. They are, however, still D'Atri spaces, i. e. all of their (locally defined) geodesic symmetries preserve the volume element. Recall that the D'Atri property is in general quite flexible [KPV], for instance, any naturally reductive homogeneous space is a D'Atri space.

However, in nonpositive curvature, we obtain the following rigidity result by combining the above results with Theorem 4.7 of [Heb2]:

Corollary 1.4. *A homogeneous space M with $K \leq 0$ is a* D'Atri space*, iff it is isometric to a Riemannian product,*

$$\mathbb{R}^k \times T^l \times M_1 \times \ldots \times M_m \times N_1 \times \ldots \times N_n \,,$$

where T^l denotes a flat torus, each M_i is irreducible symmetric of noncompact type and each N_j is a nonsymmetric Damek-Ricci space. Any of the factors may be absent.

The organization of the paper is outlined in section 2.1.

2 Proof of Theorem 1.1

We recall and comment on the definition of harmonic and asymptotically harmonic spaces.

Definition 2.1. Let (M, g) denote a complete Riemannian manifold.

(i) M is called a *harmonic space*, if about any point the geodesic spheres of sufficiently small radii are of constant mean curvature.

(ii) M is called *asymptotically harmonic*, if M has no conjugate points and the mean curvature of its horospheres is constant.

For M without conjugate points, *horospheres* in the universal covering $\widetilde{M}$ are defined as level sets of Busemann functions $b_v(x) := \lim_{t\to\infty} d(x, \gamma_v(t)) - t\,,\ x \in \widetilde{M}\,,\ v \in S\widetilde{M}$. We call $b_v^{-1}(0)$ the *stable horosphere defined by* v. As for regularity of horospheres, see remark 2.2 (c) below.

For detailed references on harmonic spaces, we refer to [Sz]. We recall some useful information.

Remark 2.2. (a) Harmonicity is equivalent to the following (which explains the terminology): Any (locally defined) *harmonic function* satisfies the mean value property. It can be rephrased in terms of infinitely many conditions on the Riemann curvature tensor and its derivatives. These conditions are explicitly computable by a recursion formula and are named after A. J. Ledger

(see e. g. [Be]). For instance, harmonic spaces are Einstein, hence real analytic in normal coordinates, and condition (i) extends to all radii except for those which correspond to conjugate points. It is well-known that the mean curvature constant of geodesic spheres in harmonic spaces depends only on the radius.

(b) If M and hence its Riemannian universal covering space $\widetilde{M}$ is harmonic, then conjugate points occur at the same distance in every direction. Hence, either $\widetilde{M}$ is *compact*, or M has *no conjugate points*. In the latter case, it follows from the Hadamard Cartan Theorem that $\widetilde{M}$ is diffeomorphic to a euclidean space and that any pair of points is joined by a unique (minimizing) geodesic.

(c) In a general simply connected, complete Riemannian manifold M without conjugate points, Busemann functions $b_v\,,\ v \in SM\,$, are known to be of regularity $C^{1,1}$. We then define the "mean curvature $m(v)$ of the stable horosphere $b_v^{-1}(0)$ at $\pi(v)$" via the *stable Jacobi tensor* along γ_v:

Given $r > 0$, consider the endomorphism fields $E_r(t) \in \mathrm{End}(\dot\gamma_v(t)^\perp)\,, t \in \mathbb{R}\,$, defined by the Jacobi equation $0 = E_r''(t) + R(\cdot, \dot\gamma_v(t))\dot\gamma_v(t) \circ E_r(t)$ ("Jacobi tensors") and the boundary conditions $E_r(0) = \mathrm{id}, E_r(r) = 0$. Note that $-E_r'(t) \circ E_r(t)^{-1}, t \in (-\infty, r)$, is the second fundamental form of a sphere about $\gamma_v(r)$. The fields $t \mapsto E_r(t)$ converge locally uniformly on $\mathbb{R}$ to the *stable Jacobi tensor* $t \mapsto E(t)\,$, as $r \to \infty$. In fact, the second fundamental forms $U_r(v) := -E_r'(0)$ converge monotonically to $U(v) := -E'(0)$ (i. e. $U_R(v) - U_r(v)$ is negative definite for $r < R$), cf. [Gr].

One then defines $m(v) := \mathrm{trace}\, U(v)$. In particular, if M has no conjugate points, then harmonicity implies asymptotic harmonicity.

If M is *asymptotically harmonic*, then $v \mapsto \mathrm{trace}\, U(v)$ is a nonnegative constant, and in particular continuous. By monotonical convergence and Dini's Theorem, the maps $v \mapsto \mathrm{trace}\, U_r(v)$ converge locally uniformly on SM, as $r \to \infty$. Hence, the tensor fields U_r converge locally uniformly to U [Heb1]. Since $U_r(v)$ coincides with the Hessian of $b_{v,r}(x) = d(x, \gamma_v(r)) - r$ at $\pi(v)$, it follows that $b_{v,r}\,, r > 0\,$, converge in the local C^2-topology to b_v, as $r \to \infty$. Hence, in this case, Busemann functions are C^2 [Esch]. Asymptotic harmonicity is equivalent to the existence of a constant $\alpha \geq 0$ such that $\Delta b = \mathrm{trace}_g \mathrm{Hess}\, b \equiv \alpha$ holds for any Busemann function $b : M \to \mathbb{R}$. See [HKS] for related discussion.

(d) Suppose that M is simply connected and asymptotically harmonic, say, $\mathrm{trace}\, U(v) \equiv \alpha \geq 0$, and that the U_r converge *uniformly* to U (e. g. if M admits a compact quotient or if M is homogeneous). Then, for any $\epsilon > 0$, there exists an $R > 0$ such that any sphere of radius at least R has mean curvatures between α and $\alpha + \epsilon$. It follows that the logarithmic volume growth rate of M equals α, that is,

$$\alpha = \lim_{r \to \infty} \frac{\log \mathrm{vol}\, B_r(p)}{r} \quad \text{for } p \in M.$$

We say that M has exponential (resp. subexponential) volume growth, if $\alpha > 0$ (resp. $\alpha = 0$).

Note that the above limit exists for arbitrary complete M, provided that M covers a compact manifold or that M is homogeneous, as was proved by A. Manning [Ma]. If $M = (G/K, \bar g)$ is an effective, Riemannian homogeneous

space with connected Lie group G, we consider the Riemannian submersion $\pi : G \to G/K, h \mapsto [h]$ where G is endowed with a suitable G-left invariant and K-right invariant metric g. Since $\pi^{-1}(B_r^{G/K}([e])) \subset B_{r+R}^G(e) \subset \pi^{-1}(B_{r+R}^{G/K}([e])$ holds for $R = \mathrm{diam}(K)$, and $\mathrm{vol}_g(\pi^{-1}(W)) = \mathrm{vol}_{\bar{g}}(W) \cdot \mathrm{vol}_g(K)$, we conclude that *subexponential growth of* G/K can be characterized in the following equivalent ways: (1) G has subexponential growth, (2) G has polynomial growth, (3) all $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{g}), X \in \mathfrak{g}$, have only purely imaginary eigenvalues (cf. sect. 6 of [Pa] for proofs and the relevant references).

2.1 Structure of the proof

We consider a simply connected homogeneous space M which is *asymptotically harmonic*. The proof of Theorem 1.1 will be split into three parts. We refer to the corresponding sections for ideas of the proofs and further details.

(i) In Proposition 2.3 we show that M admits a simply transitive solvable Lie group of isometries S. Hence, M is isometric to S, endowed with a suitable left invariant metric.

(ii) In Proposition 2.4 we show that the commutator subgroup $N := [S, S]$ has codimension 1 in S, provided that M has *exponential volume growth*. Hence, S is a semidirect product of the normal nilpotent subgroup N and a 1-dimensional complement $A \cong (\mathbb{R}, +)$.

Note that in the case of *subexponential volume growth*, S is a connected Lie group of subexponential and hence, polynomial volume growth (see remark 2.2 (d)). All $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{s}), X \in \mathfrak{s}$, have purely imaginary eigenvalues (cf. also Cor. 2.6); in particular, S is unimodular.
In the sequel, we will assume in addition that M is an *Einstein* manifold. However by [Do1], any left invariant Einstein metric on a unimodular solvable Lie group is flat and then, so is M.

(iii) In section 2.4, we recall facts about Damek-Ricci spaces. Finally, Theorem 2.9 completes the proof of Theorem 1.1: We prove that any semidirect product $S = A \cdot N$ as above, endowed with an asymptotically harmonic Einstein metric, has constant curvature, or is a Damek-Ricci space.

2.2 Transitive solvable isometry groups

We prove the following (cf. [Wo] for related discussion):

Proposition 2.3. *Every simply connected, homogeneous,* asymptotically harmonic *space M admits a simply transitive solvable group S of isometries.*

Proof. Let $G := \mathrm{Iso}_0(M)$ denote the identity component of the isometry group of M and denote by $K := G_p$ the isotropy subgroup of some point $p \in M$. Let $\hat{K} \supset K$ be a maximal compact subgroup of G. We prove that $K = \hat{K}$:

Note that $\hat{K}$ is connected; moreover, $M_0 := G/\hat{K}$ is contractible, in fact, diffeomorphic to a euclidean space (cf. Thm. 3.1 of [I] or Thm. 3.1 of ch. XV of [Ho]). I follows that the canonical $\hat{K}$-principal fibre bundle $G \to G/\hat{K} = M_0$ is

trivial (cf. Cor. 10.3 of ch. 4 of [Hu]), and that the associated projection fibre bundle $M = G/K \to G/\hat{K} = M_0$ with fibre $\hat{K}/K$ is trivial (see sect. 7 of ch. 4 of [Hu]). In particular, M is homotopy equivalent to the compact, connected manifold $\hat{K}/K$ and hence has nontrivial $\mathbb{Z}_2$-homology in dimension $n = \dim \hat{K}/K$. On the other hand, M is diffeomorphic to a euclidean space, since M is simply connected with no conjugate points (Def. 2.1). It follows that $\hat{K}/K$ is 0-dimensional. Hence, $K = \hat{K}$ is maximal compact in G.

The assertion of the proposition follows from a standard Lie group argument, if the center $Z(G)$ is trivial [Wo], and requires a geometric argument in addition, if $Z(G)$ is nontrivial:

(a) $Z(G)$ is trivial: Consider a Levi decomposition of the Lie algebra of G, say, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r} = \mathfrak{l}_c \oplus \mathfrak{l}_{nc} \oplus \mathfrak{r}$, where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}$, and $\mathfrak{l}$ denotes a semisimple complementary subgroup which we decompose into the direct sums $\mathfrak{l}_c$ and $\mathfrak{l}_{nc}$ of its compact ideals and its noncompact ideals, respectively. We denote the connected Lie subgroups, corresponding to this decomposition, by $L_c\,, L_{nc}$ and R, respectively. We also choose an Iwasawa decomposition $L_{nc} = \bar{K} \cdot A \cdot N$ of the noncompact semisimple group and recall that A is abelian and normalizes the nilpotent subgroup N, while $\bar{K}$ contains the center $\bar{Z} := Z(L_{nc})$ and $\bar{K}/\bar{Z}$ is compact.

Since L_{nc} is a linear semisimple group, $\bar{Z}$ is finite and hence $\bar{K}$ is compact. Since all maximal compact subgroups of G are pairwise conjugate in G, we may assume that K contains $L_c \cdot \bar{K}$. Hence, $\bar{S} := A \cdot N \cdot R$ is a solvable subgroup of G which acts transitively on M. Since M is simply connected, $\bar{S}$ contains a simply transitive (solvable) subgroup S, as asserted.

(b) Suppose that $Z(G)$ is nontrivial; let $\mathrm{id}_M \neq \varphi \in Z(G)$: Then, the displacement function $p \mapsto d(p, \varphi(p))$ on M is constant ("Clifford translation"), say, equal to α which is nonzero (since the isometry group acts effectively). For any $p \in M$, the geodesic $\gamma_{p\varphi(p)}$ through $p = \gamma_{p\varphi(p)}(0)$ and $\varphi(p) = \gamma_{p\varphi(p)}(\alpha)$ is φ-invariant: In fact for $0 < t < \alpha$, we have $\alpha = d(\gamma_{p\varphi(p)}(t), \varphi\,\gamma_{p\varphi(p)}(t)) \leq d(\gamma_{p\varphi(p)}(t), \varphi(p)) + d(\varphi(p), \varphi\,\gamma_{p\varphi(p)}(t)) = \alpha$ and hence the broken geodesic from $\gamma_{p\varphi(p)}(t)$ via $\varphi(p)$ to $\varphi\,\gamma_{p\varphi(p)}(t)$ is a smooth geodesic.

This provides a smooth foliation of M by φ-invariant geodesics; choose one, say, γ . For any $p \in M$, we obtain that

$$\begin{aligned} b_\gamma(p) - b_\gamma(\varphi(p)) &= \lim_{t\to\infty} (d(p, \gamma(t)) - d(\varphi(p), \gamma(t))) \\ &= \lim_{n\to\infty} (d(p, \varphi^{n+1}(\gamma(0))) - d(\varphi(p), \varphi^{n+1}(\gamma(0))) \\ &= \lim_{n\to\infty} ((n+1)\alpha + b_\gamma(p) - n\alpha - b_\gamma(p)) \\ &= \alpha = d(p, \varphi(p))\,. \end{aligned}$$

Since Busemann functions are Lipschitz with Lipschitz constant 1, it follows that the (unit length) gradient fields of b_γ and b_{γ^-} are tangent to the geodesic foliation, and hence, $b_\gamma + b_{\gamma^-}$ is constant. We conclude that $\Delta b_\gamma = -\Delta b_{\gamma^-}$. However, since M is asymptotically harmonic, it follows that $\Delta b_\gamma \equiv 0$ (cf. remark 2.2 (c)). Hence, all horospheres in M are minimal, and M has subexponential volume growth.

Now remark 2.2 (d) implies that G has polynomial growth and hence, that all $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{g}), X \in \mathfrak{g}$, have only purely imaginary eigenvalues. In particular, $\mathfrak{l}_{nc}$ is trivial. Finally, since $L_{nc} = \{\mathrm{id}_M\}$, it follows that the solvable radical R acts transitively on M. As in step (a), this completes the proof. □

2.3 The algebraic rank of S

According to section 2.2, any simply connected, homogeneous asymptotically harmonic space M admits a simply transitive solvable group of isometries S. Hence, M is isometric to S, endowed with a suitable left invariant metric $\langle\,,\rangle$ ("*solvmanifold*"). We now determine codim $[S, S]$.

Proposition 2.4. *Let $M \cong (S, \langle\,,\rangle)$ be a solvmanifold which is asymptotically harmonic with* exponential volume growth*. Then, the commutator subgroup $N := [S, S]$ has codimension 1 in S.*

We recall from (ii) of section 2.1, that asymptotically harmonic solvmanifolds of *subexponential* volume growth are *flat*, provided that the metric is Einstein.

Proposition 2.4 yields directly the following

Corollary 2.5. *$M \cong (A \cdot N, \langle\,,\rangle)$, semidirect product with $A \cong (\mathbb{R}, +)$.*

Proof of 2.4. We consider stable horospheres along geodesics perpendicular to $[S, S] \subset S$ and relate their shape operators to maximal solutions of certain algebraic Riccati equations (see (2) below). Results from control theory [LR] allow one to compute the mean curvature of the horosphere in question (see (3)). Since mean curvatures are constant by assumption, the claim follows.

Consider the metric Lie algebra $(\mathfrak{s}, \langle\,,\rangle)$ of left invariant vector fields on S and its orthogonal vector space decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ (note that $\mathfrak{a}$ need not be a subalgebra!). Choose a unit vector $A \in \mathfrak{a}$. If we decompose $\mathrm{ad}_A = D_A + S_A$ into its symmetric and skew symmetric part (w. r. t. $\langle\,,\rangle$), then Jacobi operator and covariant derivative along A are given by $R_A := R(\cdot, A)A = -D_A^2 - [D_A, S_A]$ and $\nabla_A = S_A$ on $\mathfrak{s}$, respectively (cf. e. g. [AW1]). Note that $\nabla_A A = 0$.

We compute the mean curvature of stable horospheres $H(t)$ along $\gamma_A(t)$. Their shape operators $L(t) \in \mathrm{End}(\dot\gamma_A(t)^\perp)$ satisfy the Riccati equation

$$L'(t) + L^2(t) + R(t) = 0 \quad (L(t) = L(t)^T) \tag{1}$$

(where $R(t) = R(\cdot\,, \dot\gamma_A(t))\dot\gamma_A(t)$). Since γ_A is a one-parameter subgroup of S, it follows that, in an orthonormal basis of left invariant vector fields, $L(t)$ is constant, say, $L(t) \equiv L_0 = L_0^T \in \mathrm{End}(A^\perp)$. More precisely, writing $Y(t) := Y \circ \gamma_A$ for any $Y \in \mathfrak{s}$, we obtain

$$\begin{aligned} 0 &= L'(t)Y(t) + L(t)^2 Y(t) + R(t)Y(t) \\ &= (\nabla_A(L_0 Y))(t) - (L_0 \nabla_A Y)(t) + (L_0^2 Y)(t) + (R_A Y)(t) \\ &= ([S_A, L_0]Y + L_0^2 Y - D_A^2 Y - [D_A, S_A]Y)(t)\,. \end{aligned}$$

If we write $L_0 = -D_A - X$, then we obtain that $X = X^T$ is a solution of the algebraic (matrix) Riccati equation

$$X^2 + X \circ \mathrm{ad}_A + \mathrm{ad}_A^T \circ X = 0\,. \tag{2}$$

Since any solution of (2) corresponds to a solution of (1) defined globally on $\mathbb{R}$ and L is the minimal among all such solutions (see e. g. Proposition 3' of [EO]), it follows that X is the unique *maximal solution* of (2). We recall from [LR] that for this solution, $-\mathrm{ad}_A - X$ has eigenvalues of nonpositive real parts. Now since (2) is equivalent to

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}^{-1} \begin{pmatrix} -\mathrm{ad}_A & -I \\ 0 & \mathrm{ad}_A^T \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} -\mathrm{ad}_A - X & -I \\ 0 & \mathrm{ad}_A^T + X \end{pmatrix},$$

it follows that any eigenvalue of $-\mathrm{ad}_A - X$ is also an eigenvalue of $-\mathrm{ad}_A$ or of $+\mathrm{ad}_A^T$. We conclude that

$$\operatorname{trace} L_0 = \operatorname{trace}\left(-\mathrm{ad}_A - X\right) = -\sum_\sigma \left|\mathrm{Re}(\sigma)\right|, \tag{3}$$

where the sum is taken over all eigenvalues σ of ad_A, with multiplicities.

Since $\mathfrak{s}$ is solvable, it follows from Lie's Theorem that the eigenvalues of ad_X depend linearly on $X \in \mathfrak{s}$. On the other hand, since M is asymptotically harmonic of exponential volume growth, the expression in (3) is nonzero and independent of $A \in \mathfrak{a}$, $\|A\| = 1$. This is impossible unless $\mathfrak{a}$ is one-dimensional. $\square$

The proof includes the following explicit information

Corollary 2.6. *Let $M \cong (S, \langle\,,\rangle)$ be a solvmanifold and let γ_A , $A \in \mathfrak{s}$, $\|A\| = 1$, denote a geodesic perpendicular to $[S,S]$ at e. If γ_A contains no conjugate points, then the trace of the stable Riccati solution along γ_A is equal to*

$$-\sum_\sigma \left|\mathrm{Re}(\sigma)\right|,$$

where the sum is taken over all eigenvalues σ of ad_A, with multiplicities.

2.4 Geometric characterization of Damek-Ricci spaces

In this section, we consider a simply connected homogeneous space M which is asymptotically harmonic and Einstein. We prove in Theorem 2.9 that M is isometric to a Damek-Ricci space (Def. 2.7), unless it is a space of constant sectional curvature $K \equiv c \leq 0$.

Following Proposition 2.3, M is isometric to a solvable Lie group S, endowed with a left invariant metric $\langle\,,\rangle$. By (ii) of section 2.1, either M is flat, or M has

exponential volume growth. If the latter condition is satisfied, then $N = [S, S]$ has codimension 1 in S, as follows from Proposition 2.4; we will restrict our attention to this algebraic structure.

A Lie algebra $\mathfrak{n}$ is called 2-step nilpotent, it its derived algebra $[\mathfrak{n}, \mathfrak{n}]$ lies in the center $\mathfrak{z} := \mathfrak{z}(\mathfrak{n})$. If $\mathfrak{n}$ is 2-step nilpotent and is endowed with a positive definite scalar product $\langle\, ,\rangle$, then we consider the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$ in $\mathfrak{n}$. Following A. Kaplan [Ka], the Lie bracket of $\mathfrak{n}$ defines an endomorphism

$$j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}, \langle\, ,\rangle)\,, \quad \langle j(Z)V, W\rangle := \langle [V, W], Z\rangle \quad \text{for } Z \in \mathfrak{z}\,, V, W \in \mathfrak{v}\,,$$

with values in the skew symmetric endomorphisms of $(\mathfrak{v}, \langle\, ,\rangle)$. Note that, conversely, any such endomorphism defines a 2-step nilpotent Lie bracket on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ (with center containing $\mathfrak{z}$).

Definition 2.7. (*i*) A metric 2-step nilpotent Lie algebra $(\mathfrak{n}, [,], \langle\, ,\rangle)$ is said to be *of Heisenberg type* (or *of type* H) if its j-map satisfies

$$j(Z)^2 = -\mathrm{id}_{\mathfrak{v}} \quad \text{for all } Z \in \mathfrak{z}, \|Z\| = 1.$$

(*ii*) A solvable Lie group with left invariant metric $(S, \langle\, ,\rangle)$ is called a *Damek-Ricci space* if $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is 2-step nilpotent of Heisenberg type, and $\mathfrak{n}^\perp$ is spanned by a unit vector H, such that ad_H equals $1/2 \cdot \mathrm{id}$ on $\mathfrak{v}$ and $1 \cdot \mathrm{id}$ on $\mathfrak{z}$.

Remark 2.8. Condition (*i*) implies that j extends to an algebra homomorphism between the Clifford algebra $Cl(\mathfrak{z}, \langle\, ,\rangle)$ and $\mathrm{End}(\mathfrak{v})$, i. e. Heisenberg type algebras with l-dimensional center are in 1-1-correspondence with modules over Cl_l (in the notation of [LM]). Among all Heisenberg type algebras, the Iwasawa subalgebras of the real semisimple Lie algebras of split rank one are of particular interest: The corresponding Damek-Ricci spaces are the rank-one symmetric spaces of noncompact type (and nonconstant curvature), that is, the hyperbolic spaces over $\mathbb{C}\,, \mathbb{H}$ and the hyperbolic Cayley plane.

Damek-Ricci spaces satisfy $K \leq 0$ and are Einstein [Bo], in fact, harmonic [DR]. In this class, only the rank-one symmetric spaces have strictly negative curvature [La], [Do2]. For a detailed discussion of Damek-Ricci spaces, we refer to [BTV] (for homogeneous spaces with $K \leq 0$, cf. [Hei], [Al], [AW1], [AW2]; for noncompact homogeneous Einstein spaces, cf. [Heb3]).

Rigidity results for Damek-Ricci spaces have been obtained under suitable algebraic assumptions: Suppose that S is a solvable Lie group endowed with a left invariant *harmonic* Riemannian metric g. It was proved in [BPR] and in [Dr2] (under further weak algebraic restrictions) that (S, g) is isometric to a Damek-Ricci space provided that the commutator subgroup $[S, S]$ is two-step nilpotent and of codimension one.

Our proof of 2.9 yields that the above-mentioned additional a priori algebraic assumptions on $[S, S]$ are not necessary. Moreover, asymptotic harmonicity and the Einstein condition are sufficient as the geometric requirements. The proof starts with explicit calculations of certain geodesics and Jacobi operators by M. Druetta [Dr1] and was in part inspired by those.

Theorem 2.9. *Let S denote a simply connected solvable Lie group with commutator subgroup $N = [S,S]$ of codimension 1. If S admits a nonflat, asymptotically harmonic Einstein metric $\langle\, ,\rangle$, then up to scaling of the metric, $(S, \langle\, ,\rangle)$ is constantly curved or isometric to a Damek-Ricci space.*

Proof. We consider certain geodesics γ_Z, emanating from e, tangent to the center of N $((a),(b))$. In (b), we determine all stable Jacobi fields along γ_Z explicitly in terms of hypergeometric functions. We then obtain formula (20) for the mean curvature of stable horospheres along γ_Z (cf. (c)). The formula involves a product of expressions in hypergeometric functions (19) which, by asymptotic harmonicity, has to be constant. An analytic continuation argument (cf. (d)) then shows that, in fact, every factor of this product is constant. This yields strong restrictions on the algebraic structure of S (cf. (e)), as required.

(a) We decompose the Lie algebra of S orthogonally as $\mathfrak{s} = \langle H\rangle \oplus \mathfrak{v} \oplus \mathfrak{z}$ where $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}] = \mathfrak{v}\oplus\mathfrak{z}\,,\mathfrak{z} = \mathfrak{z}(\mathfrak{n})$, and H is a unit vector (note that $\mathfrak{n} = \mathfrak{v}\oplus\mathfrak{z}$ is nilpotent, since $\mathfrak{s}$ is solvable, but not a priori 2-step).

Since $\operatorname{codim} N = 1$, it follows that $(S, \langle\, ,\rangle)$ is a noncompact homogeneous Einstein space of *standard type*, as defined in [Heb3]. According to the structure theory developed in [Heb3], the subspace $\mathfrak{v} = \mathfrak{n}\cap\mathfrak{z}^\perp$ is ad_H-invariant; moreover (possibly after passing to an isometric metric Lie algebra, see Theorem 4.10 of [Heb3]),

$$\mathrm{ad}_H : \mathfrak{v} \to \mathfrak{v}, \quad \mathrm{ad}_H : \mathfrak{z} \to \mathfrak{z}$$

are both self-adjoint with *positive* eigenvalues

$$\rho_1 \le \ldots \le \rho_k \ \text{ resp. } \mu_1 \le \ldots \le \mu_l =: \lambda$$

(possibly after replacing H by $-H$). We will not make use of the fact (cf. [Heb3]) that the eigenvalues are integers, multiplied by a constant factor.

The eigenspace of ad_H corresponding to the maximal eigenvalue lies in the center $\mathfrak{z}$ of $\mathfrak{n}$. Hence, the maximal eigenvalue equals λ, we have $\rho_k < \lambda$ and, after suitably rescaling the metric, we may and will assume that $\lambda = 1$.

We choose an arbitrary unit length eigenvector $Z \in \mathfrak{z}\,, \mathrm{ad}_H Z = \lambda Z$. As in the beginning of section 2.4, we define a skew symmetric endomorphism $j(Z) \in \mathfrak{so}(\mathfrak{v})$ by $\langle [V,\tilde V], Z\rangle_{\mathfrak{z}} = \langle j(Z)V, \tilde V\rangle_{\mathfrak{v}}$ (note that the collection of all $j(Z)\,, Z\in\mathfrak{z}$, gives only partial information about the Lie bracket, unless $\mathfrak{n}$ is 2-step nilpotent).

For later reference, we note that

$$[j(Z)^2, \mathrm{ad}_H] = 0 \quad \text{on } \mathfrak{v}\,. \tag{4}$$

In fact, if $\mathrm{ad}_H V = \rho V\,, \mathrm{ad}_H W = \rho' W\,, V, W \in \mathfrak{v}$, then $[V,W]$ lies in the $(\rho+\rho')$-eigenspace of ad_H. Hence, $\langle j(Z)V, W\rangle = \langle [V,W], Z\rangle$ vanishes unless $\rho+\rho' = 1$. It follows that $j(Z)$ maps any ρ-eigenspace of $\mathrm{ad}_H|_{\mathfrak{v}}$ to its $(1-\rho)$-eigenspace, and that $j(Z)^2$ leaves any eigenspace of ad_H on $\mathfrak{v}$ invariant.

In the sequel, we will prove that $\mu_j = 1$ holds for all j, $\rho_i = 1/2$ holds for all i, and that $j(Z)^2 = -\mathrm{id}_{\mathfrak{v}}$. Since this holds for all $Z \in \mathfrak{z}\,, \|Z\| = 1$, we find that either $\mathfrak{v} = 0\,, \mathrm{ad}_H|_{\mathfrak{z}} = \mathrm{id}_{\mathfrak{z}}$ and $(S, \langle\, ,\rangle)$ is isometric to real hyperbolic space, or

that $\mathfrak{n}$ is nonabelian, i. e. $\mathfrak{v} \neq 0$, and that $(S, \langle\, , \rangle)$ is isometric to a Damek-Ricci space. This observation will complete the proof.

(b) We recall from Lemma 1 of [Dr1] that the geodesic with initial velocity Z is given by $\gamma_Z(t) = \exp(\tanh(t)Z) \cdot \exp(-\log\cosh(t)H)$ (in terms of the Lie group exponential map of S). In terms of the left invariant vector fields H and Z on S, the velocity vector field is given by $\dot\gamma_Z(t) = (-\tanh(t)H + \frac{1}{\cosh(t)}Z)|_{\gamma_Z(t)}$.

We determine *stable* (in fact, all) *Jacobi fields* along γ_Z explicitly in terms of *hypergeometric functions* (with parameters given by ad_H and $j(Z)$) and in a suitable basis of left invariant vector fields.

Among all Jacobi fields, we identify the stable ones as those whose norm is *bounded for* $t \in [0, \infty)$. In fact, since $R(\cdot, H)H = -\mathrm{ad}_H^2$ is negative definite on $H^\perp$, we conclude that $R(t) := R(\cdot, \dot\gamma_Z(t))\dot\gamma_Z(t)$ is negative definite on $\dot\gamma_Z(t)^\perp$ for t large enough, say, $t \geq t_0$. Hence, $t \mapsto \|J(t)\|$ is convex on $[t_0, \infty)$ for any Jacobi field. Since $(S, \langle, \rangle)$ has no conjugate points, any $v \in Z^\perp$ defines unique Jacobi fields J_r along γ_Z such that $J_r(0) = v$, $J_r(r) = 0$ and the stable field $J = \lim_{r\to\infty} J_r$. Obviously, $t \mapsto \|J(t)\|$ is bounded (in fact, decreasing on $[t_0, \infty)$), and J is the only Jacobi field with initial value equal to v which is bounded on $[0, \infty)$.

We show that the vector space of stable Jacobi fields is the direct sum of three subspaces, containing $(b1)$: stable Jacobi fields which are everywhere tangent to $\langle H, Z\rangle$, $(b2)$: those which are everywhere tangent to $(\mathfrak{z} \cap Z^\perp)$, and $(b3), (b4)$: those tangent to $\mathfrak{v}$.

We calculate covariant derivatives of left invariant vector fields, using the Koszul formula $2\langle\nabla_X Y, W\rangle = -\langle X, [Y, W]\rangle - \langle Y, [X, W]\rangle + \langle W, [X, Y]\rangle$, $X, Y, W \in \mathfrak{s}$ (see e. g. section 1.3 of [Dr1]): For instance, since ad_H is self-adjoint, $\nabla_H Y = 0$ holds for all $Y \in \mathfrak{s}$. Using that $\dot\gamma_Z(t) = (-\tanh(t)H + \frac{1}{\cosh(t)}Z)|_{\gamma_Z(t)}$, a routine calculation yields the following identities which hold for all left invariant vector fields $Z^* \in \mathfrak{z}$ with $Z^* \perp Z$ and for all $V \in \mathfrak{v}$:

$$\nabla_{\dot\gamma_Z(t)} Z^* = 0\,, \quad \nabla_{\dot\gamma_Z(t)} V = \frac{-1}{2\cosh(t)}(j(Z)V)|_{\gamma_Z(t)}\,. \tag{5}$$

$(b1)$ Since $\nabla_H Z = \nabla_H H = 0$ and $\nabla_Z H = -Z$, $\nabla_Z Z = H$, it follows that H and Z span a totally geodesic 2-dimensional subalgebra of constant curvature -1. We obtain a 1-dimensional space of orthogonal stable Jacobi fields along γ_Z which are tangent to $\exp(\langle H, Z\rangle)$.

$(b2)$ Suppose that $\dim \mathfrak{z} \geq 2$. Let $Z^* \in \mathfrak{z}$, $Z^* \perp Z$, be a unit length eigenvector of ad_H, say, $\mathrm{ad}_H Z^* = \mu Z^*$. Recall that $0 < \mu \leq \lambda = 1$ and that the left invariant vector field Z^* is parallel along γ_Z (cf. (5)).

Note that $R(t) = \frac{-1}{\cosh^2(t)}(\mathrm{ad}_H + \sinh^2(t)\mathrm{ad}_H^2)$ on $(\mathfrak{z} \cap Z^\perp) \oplus \ker(j(Z))$ (cf. Prop. 1 of [Dr1]). In particular, $R(t)Z^* = \frac{-\mu - \sinh^2(t)\mu^2}{\cosh^2(t)}Z^*$. We obtain a 2-dimensional vector space of Jacobi fields of the form $t \mapsto h(t)Z^*|_{\gamma_Z(t)}$ where h is any solution of

$$0 = h''(t) + \frac{-\mu - \sinh^2(t)\mu^2}{\cosh^2(t)}h(t)$$

or, equivalently, $h(t) = e^{-\mu t} g(t)$ and

$$0 = g''(t) - 2\mu g'(t) + \frac{\mu^2 - \mu}{\cosh^2(t)} g(t)\,.$$

Substituting $g(t) = u(z(t))$ with $z = z(t) = \frac{1-\tanh(t)}{2} \in (0,1)$, we obtain

$$0 = z(1-z)u''(z) + (1+\mu-2z)u'(z) - \mu(1-\mu)u(z)\,, \tag{6}$$

the hypergeometric equation $z(1-z)f''(z) + (c-(a+b+1)z)f'(z) - abf(z) = 0$ with parameters $a = \mu\,, b = 1-\mu\,, c = 1+\mu\,.$ The hypergeometric function

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{\prod_{l=0}^{k-1}(a+l)\prod_{l=0}^{k-1}(b+l)}{\prod_{l=0}^{k-1}(c+l)} \cdot \frac{z^k}{k!} \tag{7}$$

provides a solution of (6) which is regular at $z = 0$ (cf. sect. 2.1.1 of [EMOT]). This yields a Jacobi field $t \mapsto e^{-\mu t} F(\mu, 1-\mu; 1+\mu; \frac{1-\tanh(t)}{2}) Z^*$ along γ_Z whose norm is bounded for $t \in [0,\infty)$, that is, a stable Jacobi field.

Note that a linearly independent solution of (6) is given by $u(z) = z^{-\mu}$. This yields a multiple of the Jacobi vector field $t \mapsto \cosh^{\mu}(t) Z^*$, restriction of the right invariant *Killing vector field* defined by $Z^* \in T_e S$.

($b3$) According to (4), the kernel of $j(Z) \in \mathfrak{so}(\mathfrak{v})$ is ad_H-invariant. Suppose that $\ker(j(Z))$ is nontrivial. Choose a unit length eigenvector $V^* \in \ker(j(Z))$ of ad_H, say, $\mathrm{ad}_H V^* = \rho^* V^*$. Recall that $0 < \rho^* < \lambda = 1$ and that V^* is parallel along γ_Z (cf. (5)). Moreover, $R(t)V^* = \frac{-\rho^* - \sinh^2(t)\rho^{*2}}{\cosh^2(t)} V^*$, and arguing as in ($b2$), we obtain a stable Jacobi field $t \mapsto e^{-\rho^* t} F(\rho^*, 1-\rho^*; 1+\rho^*; \frac{1-\tanh(t)}{2}) V^*$ along γ_Z.

($b4$) The orthogonal complement of $\ker(j(Z))$ in $\mathfrak{v}$ decomposes into an orthogonal direct sum of two-dimensional subspaces which are ad_H- and $j(Z)$-invariant. Any such subspace admits an orthonormal basis $\{V, \tilde V\}$ such that the matrix representations are

$$\mathrm{ad}_H = \begin{pmatrix} \rho & 0 \\ 0 & 1-\rho \end{pmatrix}, \quad j(Z) = \begin{pmatrix} 0 & -\theta \\ +\theta & 0 \end{pmatrix}, \quad 0 < \rho \le 1/2\,, \quad \theta > 0\,. \tag{8}$$

Let $V(t) := V(\gamma_Z(t))$ and $\tilde V(t) := \tilde V(\gamma_Z(t))$. We show that stable Jacobi fields along γ_Z with initial value in $\langle V, \tilde V\rangle$ are of the form $t \mapsto f(t)V(t) + g(t)\tilde V(t)$ with functions f, g given explicitly in terms of hypergeometric functions (15):

Note that $\frac{DV}{dt}(t) = \frac{-\theta}{2\cosh(t)} \tilde V(t)$ and $\frac{D\tilde V}{dt}(t) = \frac{+\theta}{2\cosh(t)} V(t)$ (cf. (5) and (8)). In particular, the left invariant vector fields V and $\tilde V$ span a parallel bundle along γ_Z. For later reference, we conclude that $\frac{D^2 V}{dt^2}(t) = \frac{+\theta\sinh(t)}{2\cosh^2(t)} \tilde V(t) - \frac{\theta^2}{4\cosh^2(t)} V(t)$ and $\frac{D^2 \tilde V}{dt^2}(t) = \frac{-\theta\sinh(t)}{2\cosh^2(t)} V(t) - \frac{\theta^2}{4\cosh^2(t)} \tilde V(t)$. On the other hand, the Jacobi operator acts on left invariant vector fields $\in \mathfrak{v}$ as $R(t) = \frac{1}{\cosh^2(t)}\{-\frac{1}{4} j(Z)^2 - \mathrm{ad}_H - \sinh^2(t)\mathrm{ad}_H^2 - \sinh(t) j(Z) \circ (1/2 \cdot \mathrm{id} - \mathrm{ad}_H)\}$ (Prop. 1 of [Dr1]), hence

leaving $\langle V, \tilde{V} \rangle$ invariant. Therefore, we obtain a 4-dimensional space of Jacobi fields along γ_Z of the form $t \mapsto f(t)V(t) + g(t)\tilde{V}(t)$ which solve

$$
\begin{aligned}
0 &= \left(\frac{D^2}{dt^2} + R(t) \right) (f(t)V(t) + g(t)\tilde{V}(t)) \\
&= \left(f''(t) + \frac{\theta}{\cosh(t)} g'(t) - \frac{\rho + \sinh^2(t)\rho^2}{\cosh^2(t)} f(t) + \frac{(\theta\rho - \theta)\sinh(t)}{\cosh^2(t)} g(t) \right) V(t) \\
&+ \left(g''(t) + \frac{-\theta}{\cosh(t)} f'(t) - \frac{(1-\rho) + \sinh^2(t)(1-\rho)^2}{\cosh^2(t)} g(t) + \frac{\theta\rho\sinh(t)}{\cosh^2(t)} f(t) \right) \tilde{V}(t).
\end{aligned}
\tag{9}
$$

In order to solve (9), consider

$$
A(t) = \tanh(t) \begin{pmatrix} \rho & 0 \\ 0 & 1-\rho \end{pmatrix}, \quad B(t) = A(t) + \frac{1}{\cosh(t)} \begin{pmatrix} 0 & -\theta \\ +\theta & 0 \end{pmatrix}.
$$

Then (9) is equivalent to either of the following two equations

$$
\begin{aligned}
0 &= \left(\frac{d}{dt} + A(t) \right) \circ \left(\frac{d}{dt} - B(t) \right) \begin{pmatrix} f \\ g \end{pmatrix} (t), \\
0 &= \left(\frac{d}{dt} + B^T(t) \right) \circ \left(\frac{d}{dt} - A(t) \right) \begin{pmatrix} f \\ g \end{pmatrix} (t).
\end{aligned}
\tag{10}
$$

The vector space of solutions $\binom{f}{g}(t)$ of (9) hence contains the 2-dimensional subspaces $\ker(\frac{d}{dt} - A(t))$ and $\ker(\frac{d}{dt} - B(t))$ and is in fact spanned by these (since any element $\binom{f}{g}(t)$ in the intersection satisfies $(B(t) - A(t))\binom{f}{g}(t) = 0$ and hence vanishes).

As in $(b2)$, $(b3)$, it is convenient to write down these solutions in terms of $z = z(t) = \frac{1-\tanh(t)}{2}$. First, $\ker(\frac{d}{dt} - A(t))$ is spanned by $t \mapsto \binom{\cosh^\rho(t)}{0} = \binom{(4z(1-z))^{-\rho/2}}{0}$ and $t \mapsto \binom{0}{\cosh^{1-\rho}(t)} = \binom{0}{(4z(1-z))^{(\rho-1)/2}}$ (these solutions correspond to right invariant *Killing vector fields* restricted to γ_Z). It remains to exhibit $\ker(\frac{d}{dt} - B(t))$:

To that end, let $u : (0,1) \to \mathbb{R}$ be any solution of the hypergeometric equation

$$
0 = z(1-z)u''(z) + (\rho - 2\rho z)u'(z) + \theta^2 u(z). \tag{11}
$$

(Only in step (d) of the proof, we will consider holomorphic extensions.) As usual, we consider the coefficients a, b, c of (11) which, in terms of ρ and θ, are given by $c = \rho$, $a + b + 1 = 2\rho$, $ab = -\theta^2$. Equivalently,

$$
0 = \cosh^{-2}(t)u''(z(t)) + 4\rho\tanh(t)u'(z(t)) + 4\theta^2 u(z(t)). \tag{12}
$$

Using (12), a straightforward calculation yields that

$$
\begin{pmatrix} f \\ g \end{pmatrix} (t) = \begin{pmatrix} -\cosh^{-\rho}(t)u'(z(t)) \\ 2\theta\cosh^{1-\rho}(t)u(z(t)) \end{pmatrix} = \begin{pmatrix} -(4z(1-z))^{\rho/2}u'(z) \\ 2\theta(4z(1-z))^{(\rho-1)/2}u(z) \end{pmatrix} \tag{13}
$$

solves $\binom{f}{g}'(t) = B(t)\binom{f}{g}(t)$ and hence, (10) resp. (9) (where we again abbreviate $z = z(t) = \frac{1-\tanh(t)}{2}$).

As solutions of (11), we use (cf. 2.8 (20), (22), 2.9 (1), (2) of [EMOT])

$$\begin{aligned}
u_1(z) &= F(a,b;c;z) \\
u_1'(z) &= \frac{ab}{c}F(a+1,b+1;c+1;z) \\
u_2(z) &= z^{1-c}F(1+a-c,1+b-c;2-c;z) \\
u_2'(z) &= (1-c)z^{-c}F(1+a-c,1+b-c;1-c;z) \\
&= (1-c)(z(1-z))^{-c}F(-a,-b;1-c;z)\,.
\end{aligned} \tag{14}$$

The solutions, given by (13), $u = u_1$ and $u = u_2$, respectively, are not bounded for $t \in [0,\infty)$, but adding suitable elements from $\ker(\frac{d}{dt} - A(t))$ we obtain bounded solutions:

We consider the following special solutions of (9) resp. (10).

$$\begin{aligned}
\begin{pmatrix} f \\ g \end{pmatrix}(t) &= B_\rho(z)\cdot C_{\rho,\theta}(z)\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ for } z = z(t) = \frac{1-\tanh(t)}{2}\,,\ v_i \in \mathbb{R}\,, \\
B_\rho(z) &= \begin{pmatrix} (4z(1-z))^{-\rho/2} & 0 \\ 0 & (4z(1-z))^{(\rho-1)/2} \end{pmatrix}, \\
C_{\rho,\theta}(z) &= \begin{pmatrix} -(4z(1-z))^{\rho}u_1'(z) & -(4z(1-z))^{\rho}u_2'(z) + 4^{\rho}(1-\rho) \\ 2\theta u_1(z) - 2\theta & 2\theta u_2(z) \end{pmatrix}.
\end{aligned} \tag{15}$$

Note that $B_\rho(z)\cdot C_{\rho,\theta}(z) \to 0$ as $z = z(t) \to 0$ resp. $t \to \infty$. This follows since $0 < \rho \le 1/2$ and from the definition of $u_i(z)$ (cf. (14)) in terms of hypergeometric functions. One uses the fact that hypergeometric functions are of the form $F(\ldots;z) = 1 + zG(\ldots;z)$ with $G(\ldots;z)$ analytic at $z = 0$ (cf. (7)). Hence, the solutions in (15) are bounded for $t \in [0,\infty)$ and, by the argument in the beginning of step (b), they provide all *stable Jacobi fields* with initial value in $\langle V, \tilde{V}\rangle$.

For later reference, consider the Wronskian $W(z) := W(u_1,u_2)(z)$ of (11), that is, $W(z) = u_1(z)u_2'(z) - u_2(z)u_1'(z) \Rightarrow W'(z) = \frac{2\rho z-\rho}{z(1-z)}W(z) \Rightarrow W(z) = const\cdot(z(1-z))^{-\rho}$; plugging in (14) and letting $z \to 0$, we obtain $const = 1-\rho$, and hence,

$$\begin{aligned}
&\det B_\rho(z)\cdot C_{\rho,\theta}(z) \\
&\quad= \ -4^{\rho}(1-\rho)\theta\cdot\sqrt{\frac{z}{1-z}}\cdot\left(\frac{u_1(z) + F(-a,-b;1-c;z) - 2}{z}\right) \\
&\quad= \ const_{\rho,\theta}\cdot e^{-t}\cdot\left(\frac{F(a,b;\rho;z) + F(-a,-b;1-\rho;z) - 2}{z}\right).
\end{aligned} \tag{16}$$

(c) Recall the Busemann function $b_Z(x) = \lim_{t\to\infty} d(x,\gamma_Z(t)) - t$ on $(S,\langle\,,\rangle)$ and the stable horosphere $b_Z^{-1}(-t)$ through $\gamma_Z(t)$. We compute its mean curvature $m(t)$ at $\gamma_Z(t)$ explicitly in terms of hypergeometric functions (20). By

asymptotic harmonicity, it then follows that a certain product, whose factors involve hypergeometric functions (19) is constant.

To that end, consider the stable Jacobi tensor $t \mapsto E(t) \in \mathrm{End}(\dot\gamma_Z(t)^\perp)$ as defined in remark 2.2 (c). For any parallel field $t \mapsto v(t)$, the stable Jacobi field with initial value $v(0)$ is given by $t \mapsto E(t)v(t)$.

Choose an orthonormal basis of $Z^\perp \in T_eS$ which consists of eigenvectors of ad_H as considered in $(b1) - (b4)$, say, H and bases $\{Z_j^*\}_j$ of $\mathfrak{z} \cap Z^\perp$, $\{V_k^*\}_k$ of $\mathfrak{v} \cap \ker(j(Z))$, and $\bigcup_i \{V_i, \tilde V_i\}$ for $\mathfrak{v} \cap \ker(j(Z))^\perp$; that is,

$$\begin{aligned} \mathrm{ad}_H Z_j^* &= \mu_j Z_j^* & \mathrm{ad}_H V_k^* &= \rho_k^* V_k^* \\ \mathrm{ad}_H V_i &= \rho_i V_i & \mathrm{ad}_H \tilde V_i &= (1-\rho_i)\tilde V_i \\ j(Z)V_i &= +\theta_i \tilde V_i & j(Z)\tilde V_i &= -\theta_i V_i \end{aligned} \tag{17}$$

where $0 < \mu_j \le 1\,,\ 0 < \rho_k^* < 1\,,\ 0 < \rho_i \le 1/2\,,\ \theta_i > 0$ for all i, j, k. We consider the matrix representation of $E(t)$ w. r. t. the parallel orthonormal frame field along γ_Z corresponding to this basis. We obtain a block matrix structure with (1×1)-blocks with entries e^{-t} (cf. $(b1)$), $e^{-\mu_j t}F(\mu_j, 1-\mu_j; 1+\mu_j; z)$ (cf. $(b2)$), $e^{-\rho_k^* t}F(\rho_k^*, 1-\rho_k^*; 1+\rho_k^*; z)$ (cf. $(b3)$) and (2×2)-blocks $E_i(t)$ for each i (cf. $(b4)$). The entries of $E_i(t)$ are coefficients of two stable Jacobi fields, written in a *parallel* frame field, while $B_{\rho_i}(z)\cdot C_{\rho_i,\theta_i}(z)$ as defined in (15) encodes two different stable fields in terms of the *left invariant* fields V_i and $\tilde V_i$. Hence, the matrices are transformed into each other by multiplication with a suitable constant invertible matrix from the right and a (time dependent) orthogonal matrix from the left. We conclude that $\det E_i(t) = \mathit{const}\cdot \det\left(B_{\rho_i}(z(t))\cdot C_{\rho_i,\theta_i}(z(t))\right)$ holds for each i. It then follows from (16) that

$$\det E(t) = \mathit{const}\cdot e^{-\nu t}\cdot h\left(\frac{1-\tanh(t)}{2}\right) \tag{18}$$

where $\nu = 1 + \sum_j \mu_j + \sum_k \rho_k^* + \sum_i 1 = \mathrm{trace}\,\mathrm{ad}_H$, and the function h is defined by

$$\begin{aligned} h(z) := &\prod_j F(\mu_j, 1-\mu_j; 1+\mu_j; z)\cdot \prod_k F(\rho_k^*, 1-\rho_k^*; 1+\rho_k^*; z) \\ &\cdot \prod_i \frac{F(a_i, b_i; \rho_i; z) + F(-a_i, -b_i; 1-\rho_i; z) - 2}{z}\,. \end{aligned} \tag{19}$$

Here, the a_i, b_i are given by $a_i + b_i + 1 = 2\rho_i \in (0,1]\,, a_ib_i = -\theta_i^2 < 0$. We may and will assume that $a_i < 0 < b_i$ and hence, that $b_i \le |a_i| < b_i + 1$.

The mean curvature $m(t)$ of the horosphere $b_Z^{-1}(-t)$ at $\gamma_Z(+t)$ is then given by

$$m(t) = -\frac{d}{dt}\log|\det E(t)| = \mathrm{trace}\,\mathrm{ad}_H - \frac{d}{dt}\log\left|h\left(\frac{1-\tanh(t)}{2}\right)\right|. \tag{20}$$

The second summand on the right hand side of (20) tends to 0, as $t \to \infty$, since $h(0) = \prod_i \left(\frac{a_ib_i}{\rho_i} + \frac{a_ib_i}{1-\rho_i}\right) \ne 0$. Since S endowed with the left invariant

metric $\langle\,,\rangle$ is, by assumption, asymptotically harmonic, it follows that $t \mapsto m(t)$ is constant equal to $\operatorname{trace}\operatorname{ad}_H$. Hence, the function h is constant on $(0,1)$ (the image of $t \mapsto \frac{1-\tanh(t)}{2}$). We conclude that $z \mapsto h(z)$ is constant.

(d) We prove that each of the factors of $h(z)$ is constant:

Recall that every hypergeometric function (and hence, every factor of $h(z)$) can be continued analytically along any path which avoids $0\,,1$ and ∞. Analytic continuation along a closed path yields another solution of the corresponding hypergeometric equation close to the endpoint.

For each factor $f(z)$ of $h(z)$, we investigate its analytic continuation $\tilde{f}(z)$ along a simple positive loop β, based at $1/2$, around 1. We will prove that either $\lim_{t\to 0}|\tilde{f}(t)| = \infty$, or $f(z) = \tilde{f}(z)$ is a polynomial. Since the product of all f's is constant (and different from 0), it then follows that all of the factors are constant functions. In step (e), we will derive restrictions on the defining parameters $\mu_j\,, \rho_k^*\,, \rho_i$ and θ_i.

Given $a, b, c \in \mathbb{R}$ with $c \notin \mathbb{Z}$, consider the hypergeometric equation

$$0 = z(1-z)\,u''(z) + (c - (a+b+1)z)\,u'(z) - ab\,u(z)$$

and its solutions $u_1(z) = F(a,b;c;z)$ and $u_2(z) = z^{1-c}F(1+a-c, 1+b-c; 2-c; z)$ (with the principal branch of z^{1-c}) which are linearly independent and uniquely defined, say, on $\mathbb{C}\backslash((-\infty,0]\cup[1,\infty))$. Recall from section 2.7.1 of [EMOT], that the analytic continuation of u_1 along β equals

$$\begin{aligned} \tilde{u}_1 &= B_{11}u_1 + B_{12}u_2\,, \quad \text{where} \\ B_{11} &= 1 - 2ie^{i\pi(c-a-b)}\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)}\,, \\ B_{12} &= -2i\pi e^{i\pi(c-a-b)}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b)\Gamma(a)}\,. \end{aligned} \tag{21}$$

Note that $B_{12} = 0$, iff at least one of the numbers $a, b, c-a, c-b$ lies in $\mathbb{Z}_0^-$.

($d1$) If $a = \mu_j\,, b = 1-\mu_j\,, c = 1+\mu_j$ and $0 < \mu_j < 1$, then $B_{12} \neq 0$. Moreover, $u_2(z) = z^{-\mu_j}$ and hence, $\lim_{z\to 0}|\tilde{u}_1(z)| = \infty$ (where $\tilde{u}_1(z)$ denotes the analytic continuation of $u_1(z)$ along β).

If $\mu_j = 1$, then $1 \equiv u_1(z) = \tilde{u}_1(z)$.

($d2$) If $a = \rho_k^*\,, b = 1-\rho_k^*\,, c = 1+\rho_k^*$ and $0 < \rho_k^* < 1$, then $\lim_{z\to 0}|\tilde{u}_1(z)| = \infty$, compare ($d1$).

($d3$) Factors of $h(z)$ of the third type are of the form

$$u(z) = \frac{F(a,b;c;z) + F(-a,-b;1-c;z) - 2}{z}\,,$$

where $0 < c \leq 1/2\,,\ a < 0 < b$ and $a+b+1 = 2c$. In order to exhibit the analytic continuation $\tilde{u}(z)$ of $u(z)$ along β, we consider all three summands of $u(z)$ separately, and find that

$$\tilde{u}(z) = A\frac{1}{z} + B\frac{1}{z^c} + C\frac{1}{z^{1-c}} + g(z)$$

where $A, B, C \in \mathbb{C}$ are explicitly computable in terms of (21) and g is analytic, say, in $\mathbb{C} \backslash ((-\infty, 0] \cup [1, \infty))$ and bounded near 0. Hence, $\lim_{z \to 0} |\tilde{u}(z)| = \infty$, unless $A = 0$ and either $B = C = 0$ or $c = 1/2$, $B = -C$. The latter conditions imply restrictions on a, b, c as follows:

If $c = 1/2$, then $a + b = 2c - 1 = 0$; it follows that

$$\begin{aligned} A &= -2ie^{i\pi(c-a-b)} \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)} - 2ie^{i\pi(1-c+a+b)} \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi(1-c))} \\ &= -2ie^{i\pi(1-c)} \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)} - 2ie^{i\pi c} \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi(1-c))} \end{aligned}$$

equals $-4\sin^2(\pi b)$ which vanishes iff $b \in \mathbb{Z}^+$; hence, $u(z) = 2\frac{F(-b,b;1/2;z)-1}{z}$ is a polynomial of degree $b-1$ (cf. (7)).

If $c \neq 1/2$, then $A = 0$ implies that a or b are integers. Since c is not an integer, it follows that $c - b = 1 - c + a$ is not an integer. But then, using the B_{12}-vanishing criterion in (21), $B = 0$ implies that a is an integer, while $C = 0$ yields that $-b$ is an integer. Finally, since $0 < a + b + 1 = 2c \leq 1$, it follows that $a = -b$, $c = 1/2$, a contradiction.

(e) As proved in (d), asymptotic harmonicity implies that every factor of $h(z)$ is constant. We conclude that factors involving ρ_k^* are not present (cf. ($d2$)), that all μ_j are equal to 1 (cf. ($d1$)), that $a_i = -1$, $b_i = +1$, $\rho_i = 1/2$ holds for all i (cf. ($d3$)) and hence, $\theta_i = \sqrt{-a_i b_i} = +1$.

We can reformulate this as follows: $\mathfrak{v} \cap \ker(j(Z))$ is trivial, ad_H equals $1 \cdot \mathrm{id}$ on $\mathfrak{z}$ and $1/2 \cdot \mathrm{id}$ on $\mathfrak{v}$, and $j(Z)^2 = -\mathrm{id}_{\mathfrak{v}}$. As explained in the end of step (a), this completes the proof. □

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